3. Elements of linear algebra

3.1 Matrices and vectors

Example: animal stocks (in a region):

3 farms: milking cows, pig- and cattle fattening

Farm №.	M	P	C
1	**34**	**2**	**14**
2	**120**	**-**	**-**
3	**150**	**40**	**30**

Rectangular scheme of numbers MATRIX

$$\Rightarrow \begin{pmatrix} 34 & 2 & 14 \\ 120 & 0 & 0 \\ 150 & 40 & 30 \end{pmatrix}$$

3.1.1 Definition of the term matrix

Definition: A (m, n)-matrix is a system of $\mathbf{m \cdot n}$ numbers $\mathbf{a_{ik}}$ (i = 1, 2,..., m; k = 1, 2, ..., n), which are organized in a rectangular scheme of m lines and n columns.

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & & & a_{2n} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{mn} \end{pmatrix}$$

The numbers in the scheme are called elements of the matrix.

Symbolism: $\underline{\mathbf{A}} = (\mathbf{a_{ij}})_{i=1 \dots m \quad j=1, \dots n} = (\mathbf{a_{ij}})_{(m,n)}$

i – row index

j – column index

Where is a_{32}?

Type ($\underline{A}$): = (m, n), m - number of rows
n - number of columns

Abbreviated, it is also possible to say $\mathbf{m \times n}$-matrix.

3.1.2 Special matrices

1) Matrix with only one column (Typ: (m, 1)): **column vector**,
   Matrix with only one row (Typ: (1, n)): **row vector**.
   Concerning vectors: elements $\rightarrow$ coordinates
   Number of elements $\rightarrow$ dimension of the vector

$$\underline{a} = \begin{pmatrix} \mathbf{a_1} \\ . \\ . \\ \mathbf{a_m} \end{pmatrix}$$ m is the dimension of a.

   Row vector $\underline{b}$ = $(b_1 \ldots b_n)$,
   (Also: $\vec{\mathbf{a}}$)

   Example!

2) **Quadratical matrix**: Matrix of the type: (n, n)

3) **Diagonal matrix**: $a_{ij} = 0$ for $i \neq j$.

$$\begin{pmatrix} \mathbf{a_{11}} & \mathbf{0} & . & . & . & \mathbf{0} \\ \mathbf{0} & \mathbf{a_{22}} & \mathbf{0} & & & . \\ . & \mathbf{0} & . & \mathbf{0} & & . \\ . & & \mathbf{0} & . & \mathbf{0} & . \\ . & & & \mathbf{0} & . & \mathbf{0} \\ \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{a_{nn}} \end{pmatrix}$$

   Main diagonal

Example!

4) $\underline{E} = (e_{ij})_{n,n}$: $e_{ij} = 0$ for i, j : $i \neq j$

$e_{ii} = 1$ for i = 1, ..., n

$\underline{E}$ is called n'th **identity matrix** (of n[th] order); in literature also $\underline{\mathbf{I}}$

The columns of the identity matrix $\underline{E}$ are called unit vectors.

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \underline{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \cdots, \underline{\mathbf{e}}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \underline{\mathbf{e}}_n = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$$

I[th] coordinate

5) Upper triangular matrix

$$\begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ 0 & a_{22} & & & & \\ . & . & a_{33} & & & . \\ . & & 0 & . & & . \\ . & & & . & . & . \\ 0 & . & . & . & 0 & a_{nn} \end{pmatrix}$$

3.1.3 Relations between matrices respectively vectors

$\underline{A} = \underline{B}$: (1) the same type.

(2) the corresponding elements are equal.

Example:

$$\begin{pmatrix} 4 & 2 & 100 \\ 10 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 100 \\ 10 & 0 & -2 \end{pmatrix};$$

$$\underline{u} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} \quad \Leftrightarrow \quad \begin{array}{lcr} u_1 & = & -2 \\ u_2 & = & 1 \\ u_3 & = & 5 \end{array}$$

$\underline{A} \leq \underline{B}$: (1) the same type.

(2) $a_{ij} \leq b_{ij}$

analogue $\underline{A} < \underline{B}$

Example!

3.1.4 Transpose of a matrix

$\underline{A} \rightarrow$ write the rows as columns $\rightarrow \underline{A}^T$ (also $\underline{A}'$)

Example!

If $\underline{A}^T = \underline{A}$, then A is a symmetric matrix ($a_{ij} = a_{ji}$).

Example!

3.1.5 Operations of matrices

Addition:

$\underline{A}$, $\underline{B}$ with type: (m, n),

A matrix $\underline{C}$ with $c_{ij} = a_{ij} + b_{ij}$ is called sum of the matrices

$\underline{A}$ u. $\underline{B}$: $\underline{A} + \underline{B}$

analogue $\underline{A} - \underline{B}$

Example!

Mulitplication of a real number with a matrix:

$\mathbf{k \in R}, \qquad \mathbf{k \cdot \underline{A} := (k \cdot a_{ij})}$

Example!

Multiplication of a row vector with a column vector (also scalar product or innerproduct):

Row vector · column vector,

Convention: $\underline{a}$ shall always be a column vector;
a row vector shall be written as $\underline{a}^T$.

Definition: The real number (scalar)

$$\mathbf{z = a_1 \cdot b_1 + \ldots + a_n \cdot b_n = \sum_{i=1}^{n} a_i \cdot b_i}$$

built by the two vectors

$$\underline{\mathbf{a}} = \begin{pmatrix} \mathbf{a_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a_n} \end{pmatrix} \quad \textbf{und} \quad \underline{\mathbf{b}} = \begin{pmatrix} \mathbf{b_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b_n} \end{pmatrix}$$

is called scalar product $\underline{a}^T \cdot \underline{b}$ of those vectors.

$$\underline{\mathbf{a}}^{\mathbf{T}} \cdot \underline{\mathbf{b}} = (\mathbf{a_1} \quad \mathbf{a_2} \quad . \quad . \quad . \quad \mathbf{a_n}) \begin{pmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b_n} \end{pmatrix}$$

Examples:

1) $$(2 \quad 0 \quad 1 \quad 6) \cdot \begin{pmatrix} 3 \\ 4 \\ -5 \\ \tfrac{1}{2} \end{pmatrix} = 2 \cdot 3 + 0 \cdot 4 + 1 \cdot (-5) + 6 \cdot \tfrac{1}{2} = 4$$

2) $$\underline{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \qquad \underline{u}^T \cdot \underline{v} = 0$$ (orthogonal)

3) We can write the equation $4x_1 + x_2 - 3x_3 = 10$

as a scalar product $\underline{a}^T \underline{x} = 10$ by using the vectors.

$$\underline{a}^T = (4 \quad 1 \quad -3) \text{ and } \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Multiplication of matrices

Concerning the scalar product: $\underline{a}^T \cdot \underline{b}$ we have:

Type ($\underline{a}^T$) = (1, **n**), Type (b) = (**n**, 1).

Now, we consider the matrices $\underline{A}$ and $\underline{B}$ as

$$\underline{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & -2 & 1 & 3 \\ 4 & -1 & 0 & 0 \end{pmatrix}, \qquad \underline{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \\ 0 & 0 \end{pmatrix}$$

with type ($\underline{A}$) = (3, 4), type ($\underline{B}$) = (4, 2)

and build all possible scalar products:

Column of $\underline{B}$ / row of $\underline{A}$	1	2
1	3	4
2	-3	4
3	2	1

Definition: Let the (m, p) matrix $\underline{A}$ and the (p,n)-matrix $\underline{B}$ (m, n, p $\in$ N) be given.

The elements of the (m, n)-matrix $\underline{\mathbf{C}} = (\mathbf{c}_{ik})_{i=1 \ldots m \quad k=1, \ldots n}$

are the scalarproducts of the i^{th} row of A and the k^{th} column of B.

The (m, n)-matrix $\underline{\mathbf{C}} = (\mathbf{c}_{ik})_{i=1 \ldots m \quad k=1, \ldots n}$ is called product $\underline{A} \cdot \underline{B}$ of the matrices $\underline{A}$ and $\underline{B}$.

Remarks:

1)

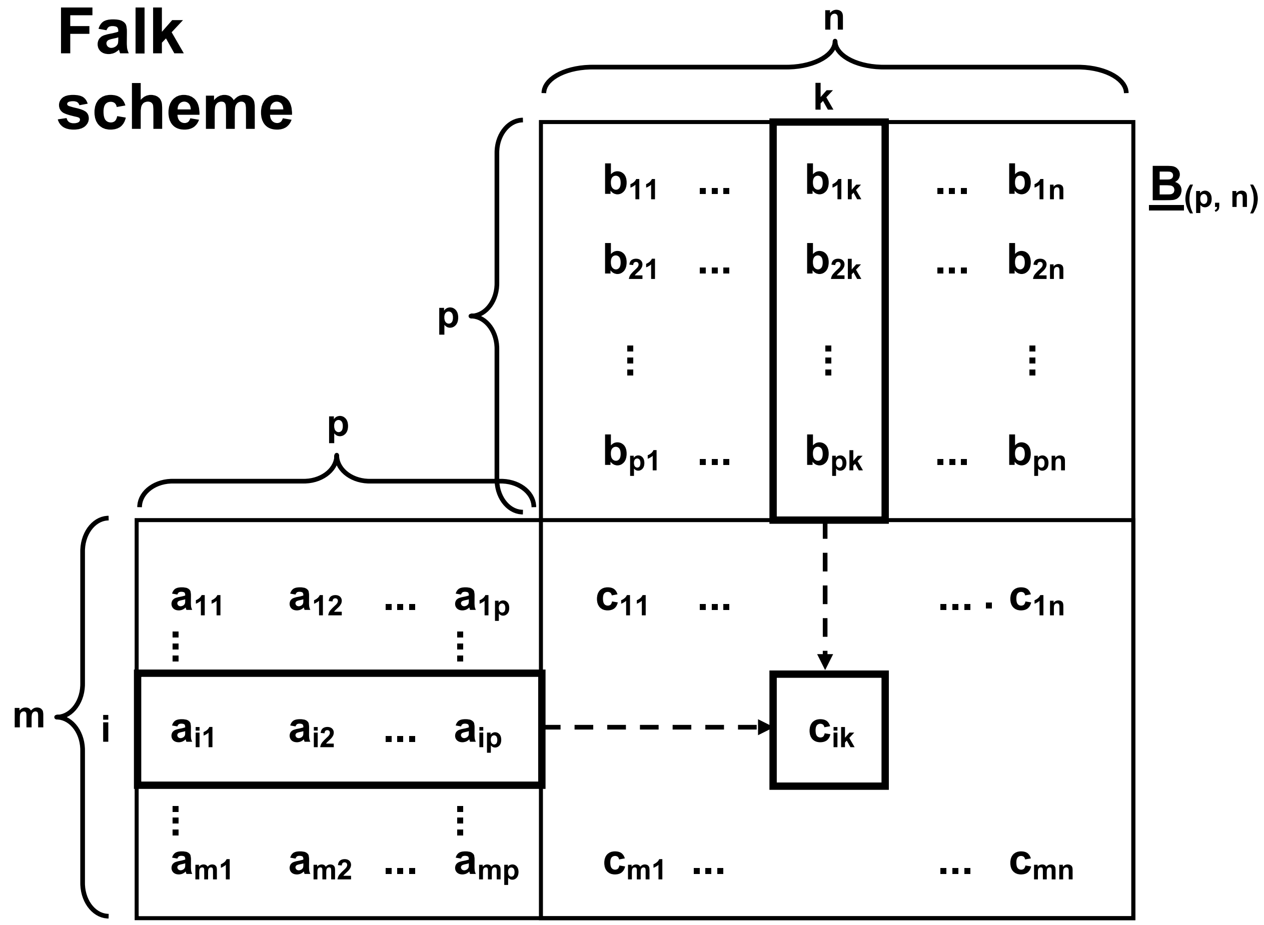
$$\underline{A}\,\underline{B} = \begin{pmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1p} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \cdot & \cdot & \cdot & b_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ b_{p1} & \cdot & \cdot & \cdot & b_{pn} \end{pmatrix}$$

$$= \left(\sum_{r=1}^{p} a_{ir} \cdot b_{rk} \right)_{i=1 \ldots m \quad k=1, \ldots n}$$

2) To calculate the product of the matrix it is recommendable to use the Falk scheme:

$$\begin{array}{cccc|cc} & & & & 1 & 0 \\ & & & & 2 & -1 \\ & & & & 1 & 2 \\ & & & & 0 & 0 \\ \hline 1 & 0 & 2 & -1 & 3 & 4 \\ 0 & -2 & 1 & 3 & -3 & 4 \\ 4 & -1 & 0 & 0 & 2 & 1 \end{array}$$

Falk scheme



$\underline{A}_{(m,\,p)}$

$\underline{C}_{(m,\,n)} = \underline{A}_{(m,\,p)} \bullet \underline{B}_{(p,\,n)}$

3) The multiplication is only possible if the number of columns of the first matrix corresponds with the number of rows of the second matrix.

4) Attention should be paid to the order of factors!

5) We have: $(\underline{A} \cdot \underline{B})^T = \underline{B}^T \cdot \underline{A}^T$

Examples:

a)

$$\begin{pmatrix} -3 & 2 & 1 \\ 0 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 1 \\ 0 & 4 & 5 \end{pmatrix}$$

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{E}} = \underline{\mathbf{A}} \;;$$

$$\underline{\mathbf{E}}_2 \cdot \underline{\mathbf{A}} = \underline{\mathbf{A}} \quad ; \quad \underline{\mathbf{E}}\underline{\mathbf{x}} = \underline{\mathbf{x}}$$

b)

$$\underline{\mathbf{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \underline{\mathbf{B}} = \begin{pmatrix} 6 & 2 \\ -3 & -1 \end{pmatrix}$$

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \underline{\mathbf{B}} \cdot \underline{\mathbf{A}} = \begin{pmatrix} 10 & 20 \\ -5 & -10 \end{pmatrix} = 5 \cdot \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

thus $\underline{A} \cdot \underline{B} \neq \underline{B} \cdot \underline{A}$

$$\underline{A} \cdot \underline{B} = \underline{0} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ but } \underline{A} \neq \underline{0} \text{ and } \underline{B} \neq \underline{0}$$

c)

$$\underline{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow$$

$$\underline{A}\,\underline{x} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & - & 2x_2 & + & x_3 \\ & & 3x_2 & + & x_3 \end{pmatrix}$$

$$\textbf{Let} \quad x^0 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \underline{A}\,\underline{x}^0 = \begin{pmatrix} -4 \\ 10 \end{pmatrix}$$

The matrix way of writing of a linear equation system.

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \end{pmatrix} \leftrightarrow \begin{array}{ccccccc} x_1 & - & 2x_2 & + & x_3 & = & -4 \\ & & 3x_2 & + & x_3 & = & 10 \end{array}$$

$\underline{A} \cdot \underline{x} = \underline{b}$ ($\underline{x}^0$ is a solution of the linear system of equations)

d) Economic example:

Raw materials $\rightarrow$ intermediate products $\rightarrow$ final products

3.1.6 Input / Output-analysis

American national economist Leontief (Beginnings: 1936-1941), separated the economy into sectors (branches, steps), material /value streams (in monetary units), „What do we need? What leaves the system?“

a) simple structure: $\boxed{\text{raw materials/ressources}} \rightarrow \boxed{\text{system}} \rightarrow \boxed{\text{products}}$

(compare to d) above)

$$\begin{pmatrix} r_1 \\ . \\ . \\ . \\ r_m \end{pmatrix} = \underline{\mathbf{R}} \cdot \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

b) Interwoven structure: e.g. relations between advanced series, production and services.

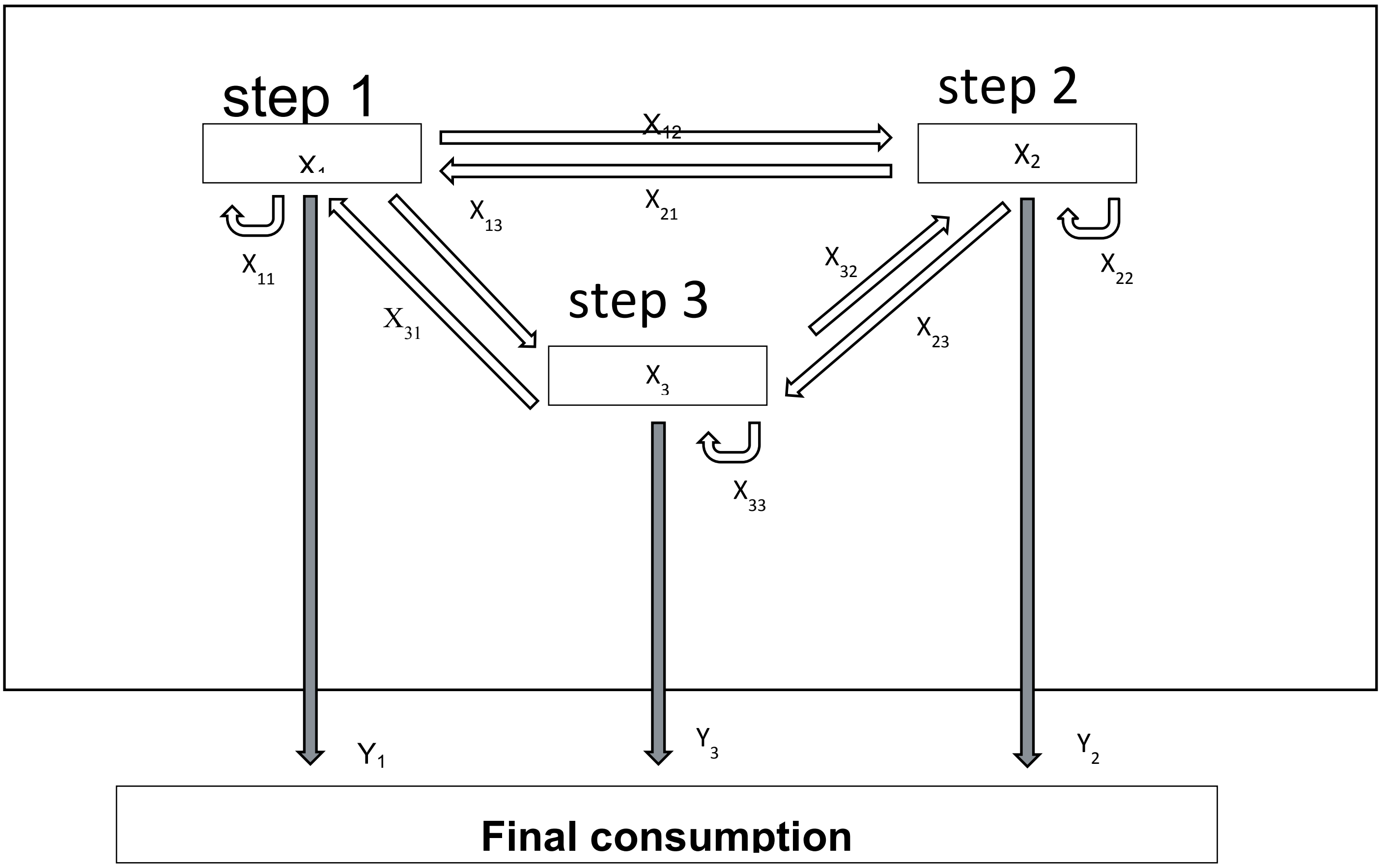
General approach n = 3:

$$\underline{\mathbf{X}} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \text{ Material flow;}$$

$$\underline{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ Final consumption vector or final demand}$$

$$\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ Production vector}$$

Economical flow sheet with three branches(respectively production branches)



accounting equations:

$$x_1 = x_{11} + x_{12} + x_{13} + y_1$$
$$x_2 = x_{21} + x_{22} + x_{23} + y_2$$
$$x_3 = x_{31} + x_{32} + x_{33} + y_3$$

$$\underline{x} = \underline{X} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \underline{y}$$

$$\mathbf{a_{ij}} = \frac{\mathbf{x_{ij}}}{\mathbf{x_j}}$$

are called input-coefficient or direct-consumption-coefficent or production coefficient.

thus

$$\underline{x} = \underline{A}\,\underline{x} + \underline{y}$$

iff. $\underline{E}\,\underline{x} = \underline{A}\,\underline{x} + \underline{y}$

iff. $\underline{E}\,\underline{x} - \underline{A}\,\underline{x} = \underline{y}$

iff. $(\underline{E} - \underline{A})\,\underline{x} = \underline{y}$

Two questions:

(1) $\underline{y}$ is given, $\underline{x}$ is wanted? $\rightarrow$ later

(2) $\underline{x}$ is given, that means that we can calculate y using the matrix-multiplication.

Example:

$$\underline{x} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 3 \\ 4 & 2 & 9 \end{pmatrix}$$ are known in a certain year.

$$\underline{A} = \left(\frac{x_{ij}}{x_j} \right) = \begin{pmatrix} \frac{1}{10} & \frac{2}{20} & \frac{6}{30} \\ \frac{3}{10} & \frac{4}{20} & \frac{3}{30} \\ \frac{4}{10} & \frac{2}{20} & \frac{9}{30} \end{pmatrix}$$

$$\underline{E} - \underline{A} = \begin{pmatrix} \frac{9}{10} & -\frac{2}{20} & -\frac{6}{30} \\ -\frac{3}{10} & \frac{16}{20} & -\frac{3}{30} \\ -\frac{4}{10} & -\frac{2}{20} & \frac{21}{30} \end{pmatrix}$$

The final consumption delivery is the result of

$$(\underline{E} - \underline{A})\ \underline{x} = \begin{pmatrix} \frac{9}{10} & -\frac{2}{20} & -\frac{6}{30} \\ -\frac{3}{10} & \frac{16}{20} & -\frac{3}{30} \\ -\frac{4}{10} & -\frac{2}{20} & \frac{21}{30} \end{pmatrix} \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 15 \end{pmatrix} \begin{matrix} \textbf{branche 1} \\ \textbf{branche 2} \\ \textbf{branche 3} \end{matrix}$$